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ON THE BRAID INDEX OF ALTERNATING LINKS

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ABSTRACT. We show that, at least for an alternating fibered link or 2-bridge link L, there is an exact formula which expresses the braid index $\mathbf{b}(L)$ of L as a function of the 2-variable generalization $P_L(l, m)$ of the Jones polynomial.

1. INTRODUCTION

Every oriented link L in S^3 is represented as a closed braid with a finite number of strings [A]. The braid index, denoted by $\mathbf{b}(L)$, of an oriented link L is then defined as the minimum number of strings needed for L to be represented as a closed braid. Obviously, the braid index is a link type invariant of L, but generally it is not easy to determine the braid index of a link.

Recently, however, some results on $\mathbf{b}(L)$ have been proved using a new polynomial $V_L(t)$ defined by V. F. R. Jones [J]. In fact, Jones finds a few criteria for L to have $\mathbf{b}(L) \leq n$ for some n [J]. Shortly after the discovery of Jones' new polynomial was announced, a number of researchers (Freyd, Yetter, Hoste, Lickorish, Millet and Ocneanu [FYHLMO] and Przytycki and Traczyk [PT]) independently introduced a 2-variable polynomial $P_L(l, m)$ as one of the many generalizations of Jones' polynomial. This polynomial $P_L(l, m)$ has a more direct connection to the braid index of L. To be more precise, let $f(l, m)$ be a 2-variable integer Laurent polynomial, i.e., $f(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$. Write $f(l, m) = \sum_{i=a}^{b} \phi_i(m) l^i$, where $\phi_a(m) \neq 0 \neq \phi_{\mathbf{b}}(m)$. We call $b - a$ the l-span of $f(l, m)$. Then it is proved [Mo] that

$$l\text{-span}\, P_L(l, m) \leq 2\{\mathbf{b}(L) - 1\}. \tag{1.1}$$

Jones observed in [J] that equality (1.1) holds for all but five knots up to 11 crossings. For alternating links it seems most likely that this is always the case, and, therefore, we propose the following conjecture.

Conjecture A. If L is an alternating (nonsplit) oriented link, then

$$l\text{-span}\, P_L(l, m) = 2\{\mathbf{b}(L) - 1\}. \tag{1.2}$$

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The main purpose of this paper is to show Conjecture A for two special but important classes of alternating links, i.e., alternating fibered links and 2-bridge links. See Theorem A and Theorem B below. Our proofs of these theorems also suggest the following slightly stronger conjecture:

Conjecture B. Let L be an oriented alternating link and $L = L_1 * L_2 * \cdots * L_h$ be a *-product representation of L, where L_i are special alternating links. (For a definition of the *-product, see §5 or [Mu2].) Then

$$(1.3) \qquad (1) \quad l\text{-span}\, P_L(l, m) = \sum_{i=1}^{h} l\text{-span}\, P_{L_i}(l, m).$$

(2) For any i, $1 \le i \le h$, $l\text{-span}\, P_{L_i}(l, m) = 2\{\mathbf{b}(L_i) - 1\}$.

(3) $\mathbf{b}(L) - 1$ is additive w.r.t. a *-product, i.e.,

$$\mathbf{b}(L) - 1 = \sum_{i=1}^{n} \{\mathbf{b}(L_i) - 1\}.$$

Obviously, Conjecture B implies Conjecture A. Furthermore, Conjecture A implies another Conjecture.

Conjecture C. Let L be an alternating oriented link. Then the exponent sum in a minimal braid representation is a link invariant. It will be denoted by $\mathbf{e}(L)$.

The implication Conjecture A $\Rightarrow$ Conjecture C follows easily from (1.2) and Corollary 2 in [Mo].

In this paper, we will prove, in fact, Conjecture B and determine $\mathbf{e}(L)$ in Conjecture C for the alternating links mentioned earlier. (See Theorem 4.2 and Proposition 6.2.) It should be noted, however, that Conjecture B no longer holds for nonalternating links, or even for positive links. For example, the (positive) trefoil knot K is a *-product of two positive Hopf links L_i. Since l-span $P_{L_i}(l, m) = 2$, we have $\sum_{i=1}^{2} l$-span $P_{L_i}(l, m) = 4$, while l-span $P_K(l, m) = 2$.

Conjectures A and B may be compared with the result [Mu4, Th] that, for an alternating link L, span $V_L(t)$ gives the minimum number of crossings among all the diagrams that L can have. This result is no longer true if L is prime and is *not* alternating.

Now we state the main theorems of this paper.

Theorem A. *Let L be an oriented alternating fibered link, and let $L = L_1 * L_2 * \cdots * L_h$ be a *-product representation of L, where each L_i is an elementary torus link. Then*

$$(1.4) \qquad (1) \quad 2\{\mathbf{b}(L) - 1\} = l\text{-span}\, P_L(l, m), \text{ and}$$

$$(2) \quad \mathbf{b}(L) = h + 1.$$

Note that $\mathbf{b}(L_i) = 2$.

Corollary 1. *If L is an oriented alternating fibered link, then the number of *-components in a reduced alternating diagram of L is an alternating-link invariant. Furthermore, the numbers of positive and negative *-components of L are also alternating-link invariants.*

A quantity $\lambda(\widetilde{L})$ associated with a link diagram $\widetilde{L}$ is called an *alternating-link invariant* if $\lambda(\widetilde{L}_1) = \lambda(\widetilde{L}_2)$ for reduced alternating diagrams $\widetilde{L}_1$ and $\widetilde{L}_2$ of the same alternating link.

If L is an alternating fibered link, then $\mathbf{b}(L)$ can be described in terms of other link type invariants.

Let $c(\widetilde{L})$ denote the number of crossings in a reduced alternating diagram $\widetilde{L}$ of L, and let $\Delta_L(t)$ be the reduced Alexander polynomial of L. Note that $c(\widetilde{L}) = \operatorname{span} V_L(t)$ [Mu4]. Then we have

Corollary 2. *Let L be an oriented alternating fibered link. Then, for any reduced alternating diagram $\widetilde{L}$ of L,*

$$\mathbf{b}(L) = c(\widetilde{L}) - \deg \Delta_L(t) + 1 . \tag{1.5}$$

Proofs of Corollaries 1 and 2 will be given in §7.

The other main theorem is

Theorem B. *Let L be a 2-bridge link of type (α, β), where $0 < \beta < \alpha$ and β is odd. Let $[2n_{1,1}, 2n_{1,2}, \dots, 2n_{1,k_1} - 2n_{2,1}, \dots, -2n_{2,k_2}, \dots, (-1)^{t-1}2n_{t,1}, \dots, (-1)^{t-1}2n_{t,k_t}]$ be a continued fraction form of $\alpha/(\alpha - \beta)$, where $n_{i,j} > 0$ for all i, j* [B-Z]. *Then*

$$\begin{aligned} &(1) \quad 2\{\mathbf{b}(L) - 1\} = l\text{-}\operatorname{span} P_L(l, m) \text{ and} \\ &(2) \quad \mathbf{b}(L) = \sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{ij} - 1) + t + 1 . \end{aligned} \tag{1.6}$$

Since proving Theorem B is much easier than proving Theorem A, we will first give a proof of Theorem B in §§3–4. A proof of Theorem A will be given in §§5–6. Generalizing Theorem B to alternating Montesinos' links is straightforward, thus we state the formula without a proof in §4. §7 contains several consequences of these theorems. In particular, we will show that a diagram of a torus link of type (q, r), $2 \le q \le r$, which is obtained from a closed q-braid representation, is a minimal diagram. See Proposition 7.5. In the last section §8, we will prove one theorem on $\mathbf{b}(L)$ for a special type of alternating links which was originally stated by K. Kobayashi [K2]. In the Appendix, we will prove a lemma needed in the proof of Theorem A.

2. Preliminaries

In this paper, all knots or links are oriented unless stated otherwise. We begin with some notation that will be used throughout this paper.

Let $f(l, m)$ be an integer Laurent polynomial on two variables l and m, i.e., $f(l, m) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$. We write $f(l, m) = \sum_{i=a}^{b} \phi_i(m) l^i$, where

$\phi_a(m) \neq 0 \neq \phi_b(m)$, $a \leq b$ and $\phi_i(m) \in \mathbf{Z}[m, m^{-1}]$. Denote by $d_{\max}\phi_i(m)$ the maximal degree of $\phi_i(m)$. We define

$$(2.1) \qquad \begin{aligned} &a_m(f) = d_{\max}\phi_a(m), \quad b_m(f) = d_{\max}\phi_b(m), \\ &a_l(f) = a, \quad b_l(f) = b, \quad \text{and} \quad l\text{-span}\, f(l, m) = b - a. \end{aligned}$$

Furthermore, let a_f and b_f denote the coefficients of $m^{a_m(f)}$ and $m^{b_m(f)}$ in $\phi_a(m)$ and $\phi_b(m)$, respectively. We denote $A_f = a_f m^{a_m(f)} l^{a_l(f)}$ and $B_f = b_f m^{b_m(f)} l^{b_l(f)}$. For convenience, we will call A_f the *minimum* term and B_f the *maximum* term, or the *extremal* terms, of $f(l, m)$.

Now, $P_L(l, m)$ is defined recursively as follows [L-M].

Let L_+, L_- and L_0 be diagrams of links which are identical except in the neighborhood of a crossing, where they look like

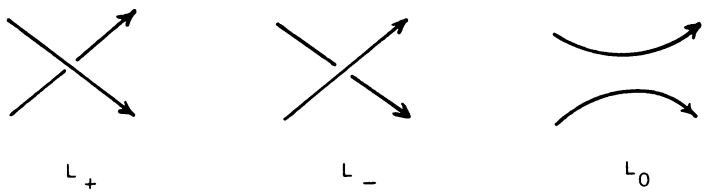


FIGURE 1

Then $P_L(l, m)$ satisfies the following formula:

$$(2.2) \qquad (1) \quad (1/l)P_{L_+}(l, m) + lP_{L_-}(l, m) + mP_{L_0}(l, m) = 0.$$

(2) If L is a trivial μ-component link, then

$$P_L(l, m) = \left(-\frac{l + l^{-1}}{m}\right)^{\mu - 1}.$$

If $f(l, m) = P_L(l, m)$, then $a_f, a_m(f), a_l(f), A_f, \ldots$ will be denoted by $a_L, a_m(L), a_l(L), A_L, \ldots$.

Our proofs of the main theorems are based upon the precise evaluations of A_L and B_L.

Let L be a link and $\tilde{L}$ a diagram of L. By changing every positive crossing $\nearrow\!\!\!\!\!\!\nwarrow$ or negative crossing $\nwarrow\!\!\!\!\!\!\nearrow$ of $\tilde{L}$ to $\asymp$, $\tilde{L}$ becomes a collection of simple closed curves on S^2, each of which is called a *Seifert circle* of $\tilde{L}$. (This operation $\times \to \asymp$ is called a *splitting*.) The number of Seifert circles in $\tilde{L}$ will be denoted by $s(\tilde{L})$.

The following proposition plays an important role in determining the braid index of a link.

Proposition 2.1 [Y]. *Let L be a link and $\tilde{L}$ its diagram. Then* $\mathbf{b}(L) \leq s(\tilde{L})$.

See [Y] for a proof.

Proposition 2.2. *Let $\mu(L)$ be the number of components of L. Then, for any diagram $\widetilde{L}$ of L, $\mu(L) \leq \mathbf{b}(L) \leq s(\widetilde{L})$.*

Proof. A link L of μ components cannot be represented as a closed braid with fewer than μ strings, and hence $\mu(L) \leq \mathbf{b}(L) \leq s(\widetilde{L})$. Q.E.D.

From (1.1), we see that, for any diagram $\widetilde{L}$ of L,

$$l\text{-span}\, P_L(l, m) \leq 2\{s(\widetilde{L}) - 1\}. \tag{2.3}$$

Proposition 2.3. *Suppose that L has a diagram $\widetilde{L}$ such that l-span $P_L(l, m) = 2\{s(\widetilde{L}) - 1\}$. Then $\mathbf{b}(L) = s(\widetilde{L})$.*

Proof. By Proposition 2.1, $\mathbf{b}(L) \leq s(\widetilde{L})$. Suppose $\mathbf{b}(L) < s(\widetilde{L})$. Say $\mathbf{b}(L) = n$. Then L has a diagram $\widehat{L}$ obtained from a natural presentation of L as a closed n-braid. Obviously, $s(\widehat{L}) = n$. Now (2.3) yields l-span $P_L(l, m) \leq 2(n-1) < 2\{s(\widetilde{L}) - 1\} = l$-span $P_L(l, m)$, a contradiction. Q.E.D.

3. Proof of Theorem B(I)

As was shown in Proposition 2.3, in order to determine $\mathbf{b}(L)$, it suffices to find a diagram $\widetilde{L}$ of a 2-bridge link L for which l-span $P_L(l, m) = 2\{s(\widetilde{L}) - 1\}$. Although L is an alternating link, the diagram we find need not be alternating.

Now the diagram we seek can be described most conveniently using the planar graph associated with a diagram. Therefore, we first recall the graph of a link diagram.

With a link diagram $\widetilde{L}$ of L, one can associate a planar graph Γ as follows. (Γ will be called the *graph* of $\widetilde{L}$ and is denoted by $\Gamma(\widetilde{L})$.) $\widetilde{L}$ divides S^2 into finitely many domains which will be classified as shaded or unshaded. No two shaded or unshaded domains have an edge in common. Each vertex corresponds to an unshaded domain and each edge of Γ corresponds to a crossing in $\widetilde{L}$. Γ is *weighted* in the sense that each edge e in Γ is assigned the weight $w(e) = +1$ or -1 as is shown in Figure 2. Γ depends on the shading ρ of the domain in $S^2 - \widetilde{L}$. If we use the different shading ρ', then the resulting graph is its dual with the opposite weight. (For the details, see [Mu6].) Conversely, given a weighted planar graph Γ on S^2, one can construct a link diagram $\widetilde{L}$ of a link L such that Γ is the graph of $\widetilde{L}$. There is, however, no way to specify the orientation of L, and, hence, the link constructed from Γ is unoriented. But, if Γ is an even graph, i.e., if every vertex has even valency, then one can specify an orientation for $\widetilde{L}$ in such a way that the boundary of a shaded domain is an *oriented* circuit. Such an oriented diagram $\widetilde{L}$ is called a *special* diagram. In fact, every oriented link has a diagram which is special [B-Z, Theorem, p. 220].

We will use a special diagram $\widetilde{L}$ of a 2-bridge link in order to evaluate A_L and B_L.

Let L be a 2-bridge link of type (α, β). Since we are only concerned about the braid index of L, we may assume w.l.o.g. that $0 < \beta < \alpha$ and β is odd.

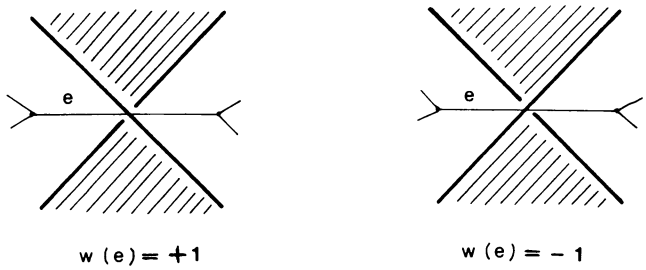


FIGURE 2

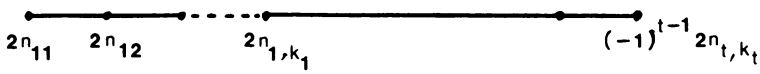


FIGURE 3

Then $\alpha/(\alpha-\beta)$ has a unique continued fraction form,

$$[2n_{11}, \dots, 2n_{1,k_1}, -2n_{2,1}, \dots, -2n_{2,k_2}, \dots, (-1)^{t-1}2n_{t,1}, \dots, (-1)^{t-1}2n_{t,k_t}],$$

where $n_{i,j} > 0$ for any i and j. Using this continued fraction from, we will construct a special diagram $\tilde{L}$ of L. We should note that if $t > 1$ then $\tilde{L}$ is not alternating.

Now, since a 2-bridge link is an algebraic link (or an *arborescent* link), a *weighted* tree T in S^2 will be associated with L. And L is the boundary of a surface F constructed by plumbing as specified by this weighted tree T. For a 2-bridge link L, T is an arc-tree, i.e., T has only two stumps. A *stump* is a vertex with valency 1. See Figure 3.

The weight at the left-end stump is $2n_{1,1}$.

Next we will define a weight $w(e) = +1$ or -1 to each edge e on T depending on whether the weight of the left-end of e is positive or negative.

Example. 2-bridge link of type (140, 61). The continued fraction form of $\frac{140}{79}$ is $[2, 4, -2, 2, 4]$.

Now take a new vertex v_0 off the tree T and join v_0 to each vertex $v_{i,j}$ (which has weight $(-1)^{i-1}2n_{i,j}$) by disjoint simple arcs. The weight of a new edge is $+1$ or -1 according to whether $(-1)^{i-1}2n_{i,j} > 0$ or not. The number, $s_{i,j}$, of edges joining v_0 and $v_{i,j}$ is determined by the following formulas.

(3.1) (I) If $t = 1$ and $k_1 = 1$, then $s_{11} = 2n_{11}$.

(II) Suppose $t \geq 2$ or $k_1 \geq 2$, then

(1) $s_{11} = 2n_{11} - 1$.

(2) Let $(i, j) \neq (1, 1)$ or (t, k_t).

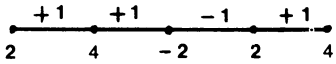


FIGURE 4

(i) If $v_{i,j}$ is the common end of two edges having the same weight, then $s_{i,j} = 2n_{i,j} - 2$.

(ii) If $v_{i,j}$ is the common end of two edges having different weights, then $s_{i,j} = 2n_{i,j}$.

$$(3) \quad \begin{aligned} s_{t,k_t} &= 2n_{t,k_t} - 1 \quad \text{if } k_t \geq 2, \text{ and} \\ s_{t,k_t} &= 2n_{t,k_t} + 1 \quad \text{if } k_t = 1. \end{aligned}$$

Now the resulting weighted graph Γ (ignoring the weights of vertices) is what we wish to use as the graph of a (special) diagram of L. See Figure 5.

Since Γ is an even graph, one can construct an oriented link diagram $\widetilde{L}$ of some link L such that Γ is the graph of $\widetilde{L}$. $\widetilde{L}$ is a special diagram of a 2-bridge link L of type (α, β). From our construction, we easily see

Proposition 3.1.

(1) *For every vertex* $v_{i,j}$ *(except for* v_0*), the weight of* $v_{i,j}$ $(= (-1)^{i-1}2n_{i,j})$ *is equal to the sum of all weights of edges incident to* $v_{i,j}$.

(2) $s_{i,j} \equiv 0 \pmod 2$ *for* $(i, j) \neq (1, 1)$ *or* (t, k_t) *and* $s_{11} \equiv s_{t,k_t} \equiv 1 \pmod 2$.

Using the special diagram $\widetilde{L}$ of a 2-bridge link L, we will determine A_L and B_L.

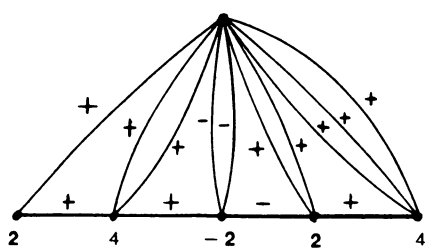


FIGURE 5

Proposition 3.2.

(3.2)

(1) $a_m(L) = \sum_{i=1}^{t} k_t - 2\left[\frac{t}{2}\right]$.

(2) $a_l(L) = -\sum_{i\equiv 0(2)} \sum_{j=1}^{k_i} 2n_{i,j} - 2\left[\frac{t}{2}\right] + \sum_{i=1}^{t} k_i$.

(3) $b_m(L) = \sum_{i=1}^{t} k_i - 2\left[\frac{t+1}{2}\right]$.

(4) $b_l(L) = \sum_{i\equiv 1(2)} \sum_{j=1}^{k_i} 2n_{i,j} + 2\left[\frac{t+1}{2}\right] - \sum_{i=1}^{t} k_i$.

(5) $|a_L| = |b_L| = 1$.

(6) $\frac{1}{2}\{b_l(L) - a_l(L)\} = \sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{i,j} - 1) + t$.

Proof. Since (6) follows from (1)–(5), we will prove only (1)–(5). A proof is an easy induction on t and k_i. Thus we will omit most of our computations. In fact, if we apply a *crossing* change $\nearrow\!\!\!\searrow \to \nearrow\!\!\!\searrow$ or $\nearrow\!\!\!\searrow \to \nearrow\!\!\!\searrow$ and *splitting* $\nearrow\!\!\!\searrow \to \asymp$ on a crossing on $\tilde{L}$, the resulting link is also a 2-bridge link with a shorter continued fraction form. Therefore, we can apply the induction assumption on the resulting links. Then the formula (2.2) will prove the proposition

Case 1. $t = 1$. If $k_1 = 1$, then L is a torus link of type $(2n_{1,1}, 2)$ and, hence, $A_L = -ml$ and $B_L = (-1)^{n_{11}-1} m^{-1} l^{2n_{11}+1}$, which proves (3.2).

If $k_1 > 1$, apply a crossing change and splitting on the crossing on the left-most edge joining v_0 and v_{11}. Then the link L_- and L_0 have, respectively, the following continued fraction forms:

$$L_-: [2n_{13}, 2n_{14}, \dots, 2n_{1,k_1}] \quad \text{if } n_{11} = 1,$$
$$[2(n_{11} - 1), 2n_{12}, \dots, 2n_{1,k_1}] \quad \text{if } n_{11} \geq 1;$$
$$L_0: [2n_{12}, 2n_{13}, \dots, 2n_{1,k_1}].$$

Since $-P_L(l, m) = l^2 P_{L_-}(l, m) + lmP_{L_0}(l, m)$, a simple computation shows:

(3.3)

(1) If $n_{11} = 1$, then $A_L = -mlA_{L_0} = \pm m^{k_1} l^{k_1}$ and
$B_L = -lmB_{L_0} = \pm m^{k_1-2} l^{N_1-k_1+2}$,

(2) If $n_{11} \geq 2$, then $A_L = -lmA_{L_0} = \pm m^{k_1} l^{k_1}$ and
$B_L = -l^2 B_{L_-} = \pm m^{k_1-2} l^{N_1-k_1+2}$,

where $N_1 = 2n_{11} + \cdots + 2n_{1,k_1}$. Now (3.3) proves (3.2) for the case $t = 1$.

Case 2. $t \geq 2$ and $t \equiv 0 \pmod 2$.

We apply a crossing change and splitting on the crossing on the rightmost edge e joining v_0 and v_{t,k_t}. We state only the continued fraction forms for L_+ and L_0. The rest of the proof consists of straightforward computations using the induction assumptions.

(1) Suppose $k_t = 1$. Then

$$L_+ : [2n_{11}, \dots, 2n_{1,k_1}, \dots, (-1)^{t-2}2n_{t-1,1}, \dots, (-1)^{t-2}2n_{t-1,k_{t-1}-1}], \quad \text{if } n_{t,1} = 1,$$

$$[2n_{11}, \dots, 2n_{1,k_1}, \dots, (-1)^{t-2}2n_{t-1,1}, \dots, (-1)^{t-2}2n_{t-1,k_{t-1}}, (-1)^{t-1}2(n_{t,1}-1)], \quad \text{if } n_{t,1} \geq 2;$$

$$L_0 : [2n_{11}, \dots, 2n_{1,k_1}, \dots, (-1)^{t-2}2n_{t-1,1}, \dots, (-1)^{t-2}2n_{t-1,k_{t-1}}],$$

and

$$A_L = -l^{-2}A_{L_+} \quad \text{and} \quad B_L = -l^{-1}mB_{L_0}.$$

(2) Suppose $k_t \geq 2$. Then

$$L_+ : [2n_{11}, \dots, 2_{n_1,k_1}, \dots, (-1)^{t-1}2n_{t,1}, \dots, (-1)^{t-1}2n_{t,k_t-2}], \quad \text{if } n_{t,k_t} = 1,$$

$$[2n_{11}, \dots, 2n_{1,k_1}, \dots, (-1)^{t-1}2n_{t,1}, \dots, (-1)^{t-1}2(n_{t,k_t}-1)], \quad \text{if } n_{t,k_t} \geq 2;$$

$$L_0 : [2n_{1,1}, \dots, 2n_{1,k_1}, \dots, (-1)^{t-1}2n_{t,1}, \dots, (-1)^{t}2n_{t,k_t-1}].$$

If $n_{t,k_t} = 1$, then $A_L = -l^{-1}mA_{L_0}$ and $B_L = -l^{-1}mB_{L_0}$. If $n_{t,k_t} \geq 2$, then $A_L = -l^2A_{L_+}$ and $B_L = -l^{-1}mB_{L_0}$.

Case 3. $t \geq 2$ and $t \equiv 1 \pmod 2$. Since a proof is analogous to Case 2, a proof will be omitted.

A proof of Proposition 3.2 is now complete.

4. Proof of Theorem B(II)

By Propositions 2.3 and 3.2(6), Theorem B will follow from Proposition 4.1 below.

Proposition 4.1. *There is a diagram $\widetilde{L}$ of a 2-bridge link L which contains exactly $\sum_{i=1}^{t}\sum_{j=1}^{k_i}(n_{ij}-1)+t+1$ Seifert circles.*

Proof. Let $\widetilde{L}$ be the diagram of L constructed in §3. Since $\widetilde{L}$ is a special diagram, the number $s(\widetilde{L})$ of Seifert circles in $\widetilde{L}$ is exactly the number of domains $S^2 - \Gamma(\widetilde{L})$. Let V and E denote, respectively, the number of vertices

and edges in Γ. Then we see easily that

$$(4.1) \qquad (1) \quad V = \sum_{i=1}^{t} k_i + 1,$$

$$(2) \quad E = \sum_{i=1}^{t} \sum_{j=1}^{k_i} (2n_{ij} - 2) + \sum_{i=1}^{t} k_i + 2t - 1.$$

Therefore, Euler's formula yields $s(\widetilde{L}) = E - V + 2 = 2\{\sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{ij} - 1) + t\}$. By comparing $s(\widetilde{L})$ with l-span $P_L(l, m) = 2\{\sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{ij} - 1) + t\}$, we must *eliminate* $\sum_{i=1}^{t} \sum_{j=1}^{k_i} (n_{ij} - 1) + t - 1$ Seifert circles from $\widetilde{L}$.

Let s_{ij} be the number of edges joining v_{ij} and v_0 defined in (3.1). Then we can eliminate $[s_{ij}/2]$ Seifert circles by simple deformation depicted in Figure 6 (which replace short arcs $\overline{P_i P_i'}$ by long arcs).

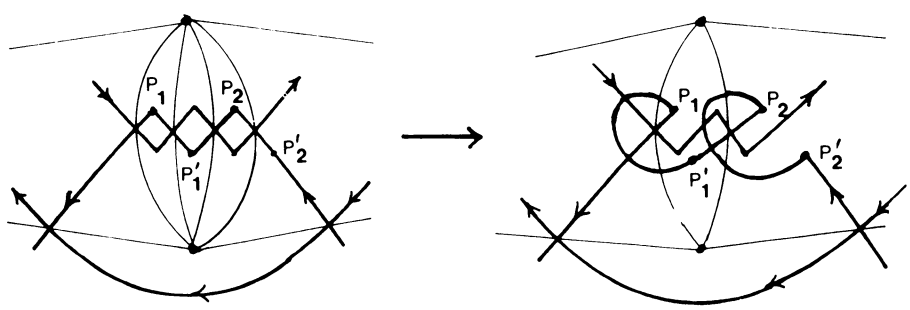
Repeated applications of this deformation eventually eliminate $\sum_{i,j} [s_{ij}/2] = \sum_{i,j} (n_{ij} - 1) + t - 1$ Seifert circles. This proves Proposition 4.1 and hence Theorem B. Q.E.D.

Theorem B and Theorem 1 in [Y] show that the exponent sum $\mathbf{e}(L)$ in a minimal braid representation of a 2-bridge link L is equal to the writhe $w(\widetilde{L})$ $(= c_+(\widetilde{L}) - c_-(\widetilde{L}))$ of a diagram $\widetilde{L}$ with a minimal number of Seifert circles, where $c_+(\widetilde{L})$ and $c_-(\widetilde{L})$ denote, respectively, the number of positive and negative crossings in $\widetilde{L}$. Therefore, it is easy to determine $\mathbf{e}(L)$ for a 2-bridge link.

Theorem 4.2. *Let L be a 2-bridge link of type (α, β). Then*

$$(4.2) \qquad \mathbf{e}(L) = \sum_{i \equiv 1(2)} \sum_{j=1}^{k_i} n_{ij} - \sum_{i \equiv 0(2)} \sum_{j=1}^{k_i} n_{ij} + \frac{1 - (-1)^t}{2}.$$

Proof. A diagram of L with a minimal number of Seifert circles is obtained from a special diagram $\widetilde{L}$ by applying isotopic deformations shown in Figure 6.



(a) (b)

FIGURE 6

If $\overline{P_iP_i'}$ is an over-passing arc, then this deformation decreases $w(\widetilde{L})$ by one. On the other hand, if $\overline{P_iP_i'}$ is an under-passing arc, then it increases $w(\widetilde{L})$ by one. Since the original diagram $\widetilde{L}$ has exactly $\sum_{i\equiv 1(2)}\{\sum_{j=1}^{k_i}(2n_{ij}-2)+k_i+2\}-1$ positive crossings and $\sum_{i\equiv 0(2)}\{\sum_{j=1}^{k_i}(2n_{ij}-2)+k_i+2\}$ negative crossings, the final diagram $\widehat{L}$ we obtained in Proposition 4.1 has the writhe

$$\begin{aligned} w(\widehat{L}) = &\sum_{i\equiv 1(2)}\left\{\sum_{j=1}^{k_i}(2n_{ij}-2)+k_i+2\right\} \\ &-1-\left\{\sum_{i\equiv 1(2)}\sum_{j=1}^{k_i}(n_{ij}-1)+\left[\frac{t-1}{2}\right]\right\} \\ &-\sum_{i\equiv 0(2)}\left\{\sum_{j=1}^{k_i}(2n_{ij}-2)+k_i+2\right\}+\sum_{i\equiv 0(2)}\sum_{j=1}^{k_i}(n_{ij}-1)+\left[\frac{t}{2}\right] \\ = &\sum_{i\equiv 1(2)}\sum_{j=1}^{k_i}n_{ij}+2\left[\frac{t+1}{2}\right]-1-\left[\frac{t-1}{2}\right]-\sum_{i\equiv 0(2)}\sum_{j=1}^{k_i}n_{ij}-\left[\frac{t}{2}\right]. \end{aligned}$$

Since $2[\frac{t+1}{2}]-[\frac{t}{2}]-[\frac{t-1}{2}]-1=(1-(-1)^t/2)$, we obtain (4.2).

The braid index of an alternating Montesinos link L is also determined without difficulty. In fact, L is an algebraic link which is associated with a weighted tree T. We state the following theorem without a proof.

Theorem 4.3. *Let T be a weighted tree associated with a Montensinos link L. Suppose that T is positive and excessive. In other words, for each vertex v of T, the weight $w(v)$ is positive and $w(v)$ is at least equal to the valency,* $\mathrm{val}(v)$ *at v. Then*

$$\mathbf{b}(L)=\sum\left[\frac{w(v)-\mathrm{val}(v)}{2}\right]+s, \tag{4.3}$$

where the summation runs over all vertices in T and s denotes the number of stumps in T.

Remark 4.1. A link in Theorem 4.3 is a special alternating link. The braid index of an alternating Montesinos link associated with a excessive but not necessarily positive tree is also determined without much difficulty. However, the braid index of a general alternating algebraic link is not easy to determine, although some appropriate algorithm may exist. We do not pursue this problem any further in this paper. Rather, we will return to a proof of Theorem A in the next section.

5. Proof of Theorem A(I)

A proof of Theorem A is much more complicated than that of Theorem B. Contrary to a 2-bridge link, it is unlikely that an alternating fibered link has

a neat diagram which reduces to a *simpler* alternating fibered link after one crossing change. Therefore, we must use a "global" argument, instead of a local one.

First we recall the definition of the *-product of two links. Let L be an oriented link and $\widetilde{L}$ its diagram in S^2. A Seifert circle C in $\widetilde{L}$ is classified as type (I) or type (II). C is said to be of type (II) if both the interior and the exterior of C contain other Seifert circles. Otherwise C is of type (I). If there are no Seifert circles of type (II), the diagram is special. Suppose that there is a Seifert circle C of type (II) in $\widetilde{L}$. If we *erase* the interior of C, we obtain a new link diagram of some link, say L_1. See Figure 7.

Example. $\widetilde{L}$ has two Seifert circles C and C', and L is a *-product of three links L_1, L_2 and L_3, i.e., $L = L_1 * L_2 * L_3$.

Similarly, by erasing the part of $\widetilde{L}$ which is exterior to C, we will obtain the other link diagram $\widetilde{L}_2$ of a link L_2. We say then that the link L is a (planar) *-product (or a (planar) Murasugi sum) of two links L_1 and L_2, written as $L_1 * L_2$. Also the diagram $\widetilde{L}$ is called a (planar) *-product of two diagrams $\widetilde{L}_1$ and $\widetilde{L}_2$, written as $\widetilde{L} = \widetilde{L}_1 * \widetilde{L}_2$. We can repeat the same argument as long as a diagram has Seifert circles of type (II) and L will be decomposed into a finite number of links $L_1, L_2, \dots, L_k$ with special diagrams $\widetilde{L}_1, \widetilde{L}_2, \dots \widetilde{L}_k$. We write $L = L_1 * L_2 * \cdots * L_n$ and call it a (planar) *-products representation of L. Similarly, a diagram $\widetilde{L}$ is written as $\widetilde{L} = \widetilde{L}_1 * \widetilde{L}_2 * \cdots * \widetilde{L}_k$. (For more details, see [Mu2 or G].) Since the connected sum of two links can be considered a planar *-product of these links, we may assume that L_i is prime. A *-product representation of a link is not unique. Nevertheless, this concept is particularly important in the study of alternating links. In fact, if L is an alternating link, each component L_i of a (planar) *-product representation of L is a special alternating (either positive or negative) link.

Now suppose that L is an alternating fibered link. Since a special alternating prime fibered link is an elementary torus link [Mu1], L is a (planar) *-product of elementary torus links, i.e., torus links of type $(q_i, 2)$ or $(q_i, -2)$, where $q_i \geq 2$.

Recall $c_+(\widetilde{L})$ and $c_-(\widetilde{L})$ to denote the number of positive and negative crossings in $\widetilde{L}$. Therefore, $c(\widetilde{L}) = c_+(\widetilde{L}) + c_-(\widetilde{L})$. Write $w(\widetilde{L}) = c_+(\widetilde{L}) - c_-(\widetilde{L})$ and $s(\widetilde{L})$ is the number of Seifert circles in $\widetilde{L}$. $h_+(\widetilde{L})$ and $h_-(\widetilde{L})$ denote the number of positive and negative *-components in $\widetilde{L}$, and hence $h(\widetilde{L}) = h_+(\widetilde{L}) + h_-(\widetilde{L})$. Then Theorem A will follow from the following

Proposition 5.1. *If L is an alternating fibered link, then*

$$(5.1) \qquad (1) \quad A_L = (-1)^{c+h_+} m^{c-h-2h_-} l^{w-h},$$
$$(2) \quad B_L = (-1)^{c+h_-} m^{c-h-2h_+} l^{w+h},$$

where, for simplicity, we delete $\widetilde{L}$ from the notation.

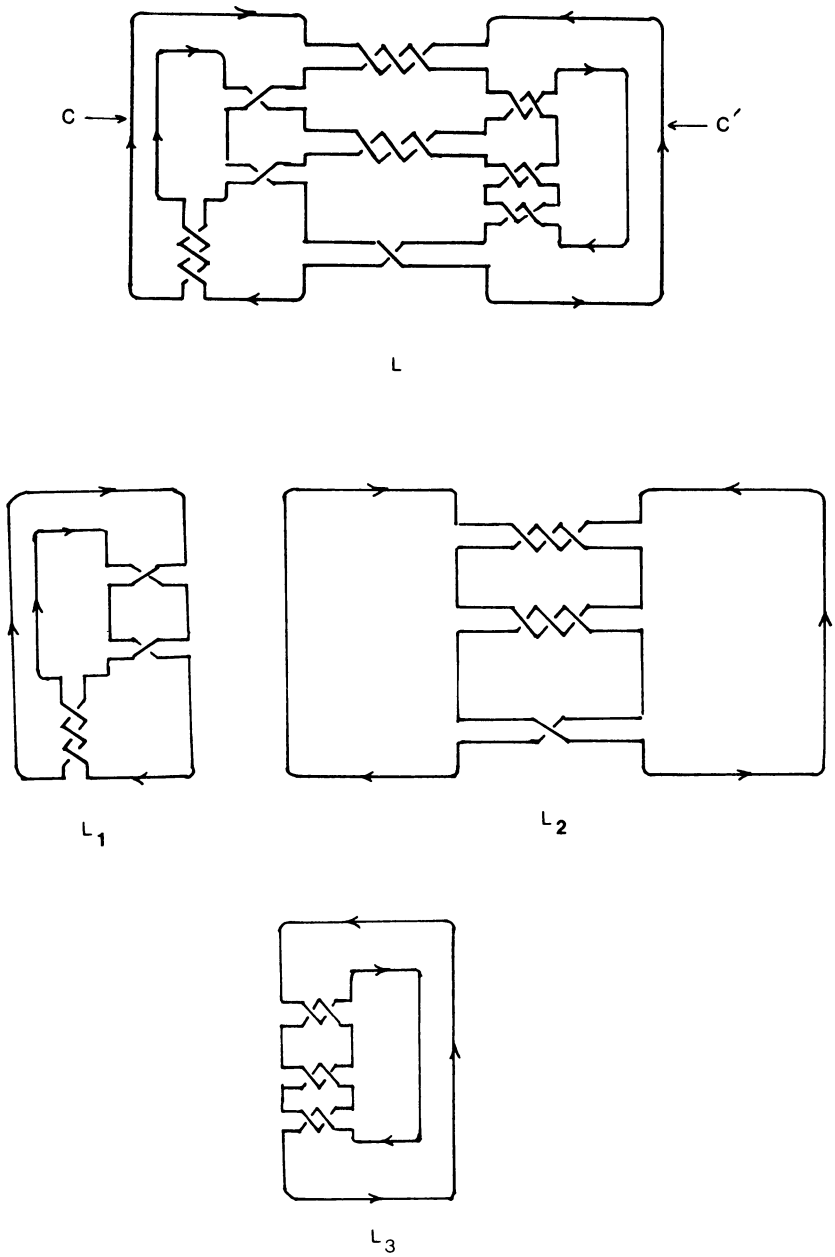


FIGURE 7

In fact, since $s(\tilde{L}) = h(\tilde{L}) + 1$, we see that l-span $P_L(l, m) = 2h(\tilde{L}) = 2\{s(\tilde{L}) - 1\}$ and hence, $\mathbf{b}(L) = s(\tilde{L}) = h(\tilde{L}) + 1$ by Proposition 2.3. This proves Theorem A.

Now, for the mirror image L^* of L, we know that $P_{L^*}(l, m) = P_L(l^{-1}, m)$ [L-M] and hence A_L is obtained immediately from B_{L^*}. Therefore, we need only prove (5.1)(2).

A proof of (5.1)(2) will be given by induction on $h(\widetilde{L})$ and $c(\widetilde{L})$.

Suppose $h(\widetilde{L}) = 1$. Then we may assume that $h_+(\widetilde{L}) = 1$ and $h_-(\widetilde{L}) = 0$. L is a torus link of type $(q, 2)$, $q \geq 2$. Then a simple computation [M-L] shows that $B_L = (-1)^q m^{q-3} l^{q+1}$. Since $c(\widetilde{L}) = q$, $h_+(\widetilde{L}) = 1 = h(\widetilde{L})$, $h_-(\widetilde{L}) = 0$ and $w(\widetilde{L}) = q$, (5.1)(2) follows immediately.

Now suppose $h \geq 2$. Suppose furthermore that all *-components are positive. Then $h(\widetilde{L}) = h_+(\widetilde{L})$ and $h_-(\widetilde{L}) = 0$. Since $\widetilde{L}$ is an alternating diagram, L must be a connected sum of these positive elementary torus links L_i. (We need the assumption here that L be alternating.) Then it is known [L-M] that $B_L = \prod_{i=1}^h B_{L_i}$, and hence we obtain (5.1)(2).

Therefore, we may assume in the rest of this section and the next that

$$h_+(\widetilde{L}) \neq 0 \neq h_-(\widetilde{L}). \tag{5.2}$$

Now, given a link L and its diagram $\widetilde{L}$, there are, say, k crossings c_1, $c_2, \dots, c_k$ on $\widetilde{L}$ such that crossing changes at these crossings make L a trivial link L_0. Then the polynomial $P_L(l, m)$ of L is given by the formula [Mu3].

$$\begin{aligned} P_L(l, m) &= (-1)^k l^{2(y-x)} P_{L_0}(l, m) \\ &\quad + (-1) \sum_{d=1}^{k} m^d \sum_{1 \leq j_1 < \cdots < j_d \leq k} l^{d-2u_d} P_{L_{j_1 \cdots j_d}}(l, m), \end{aligned} \tag{5.3}$$

where y and x are, respectively, the number of positive and negative crossings among $\{c_1, \dots, c_k\}$, u_d is the number of negative crossings among $\{c_{j_1}, \dots, c_{j_d}\}$, and $L_{j_1 j_2 \cdots j_d}$ is a link diagram obtained from $\widetilde{L}$ by splitting each of crossings $c_{j_1}, \dots, c_{j_d}$. We can choose these k crossings so that each link diagram $L_{j_1 \cdots j_d}$ is connected. If L is an alternating fibered link, then, to each $\{j_1, \dots, j_d\}$, $L_{j_1 \cdots j_d}$ is an alternating fibered link diagram or a trivial *knot*. It cannot be a trivial *link*, since $L_{j_1 \cdots j_d}$ is alternating and connected. We should note that $L_{j_1 \cdots j_d}$ may not be reduced, i.e., it may contain removable (or nugatory) crossings.

Now, to prove (5.1)(2), we carefully choose a set of crossings on which crossings changes and splittings are applied.

Let $\widetilde{L}$ be *-product of elementary torus link diagrams $\widetilde{L}_i$ of type $(q_i, 2)$ or $(q_i, -2)$, $q_i \geq 2$.

Lemma 5.2. *There are, say, k crossings $c_1, c_2, \dots, c_k$ such that* (1) *crossing changes at $c_1, c_2, \dots, c_k$ make $\widetilde{L}$ a trivial link, and* (2) *for each i, $i = 1, 2, \dots, h$, the number α_i of crossing changes applied on $\widetilde{L}_i$ is at least one but at most $q_i - 1$, i.e., $0 < \alpha_i < q_i$. In other words, each $\widetilde{L}_i$ has at least one crossing in $\{c_1, c_2, \dots, c_k\}$, but it never has all crossings in $\{c_1, c_2, \dots, c_k\}$.*

Note that if $\alpha_i = q_i - 1$, then a diagram of some $L_{j_1 \cdots j_d}$ is not reduced. Also we may not need to apply crossing changes at all crossings $c_1, \ldots, c_k$ to make $\tilde{L}$ trivial. In other words, we apply some unnecessary crossing changes to L to prove Lemma 5.2. We postpone a proof of Lemma 5.2 to the Appendix.

Now, we proceed to a proof of Proposition 5.1. Let $\{c_1, c_2, \ldots, c_k\}$ be a set of crossings found in Lemma 5.2. Consider (5.3). Write $C = l^{2(y-x)} P_{L_0}(l, m)$ and $f(l, m) = \sum_{d=1}^{k} m^d \sum_{1 \le j_1 < \cdots < j_d \le k} l^{d-2u_d} P_{L_{j_1 \cdots j_d}}(l, m)$. Then we claim

Proposition 5.3.

$$B_f = (-1)^{c+h_-} m^{c-h-2h_+} l^{w+h}. \tag{5.4}$$

A proof of Proposition 5.3 will be given in the next section. In the rest of this section, we will prove, using (5.4), that B_f is in fact the maximum term of $P_L(l, m)$.

Proposition 5.4. $B_f = B_L$. *In other words,* B_C *cannot be the maximum term of* $P_L(l, m)$.

Proof. Let μ be the number of components in L. Since L_0 is a trivial μ-component link,

$$P_{L_0}(l, m) = (-(l + l^{-1})/m)^{\mu-1}$$

and hence

$$C = l^{2(y-x)}(-(l + l^{-1})/m)^{\mu-1}.$$

Therefore, we have

$$B_C = (-1)^{\mu-1} m^{-(\mu-1)} l^{2(y-x)+(\mu-1)}. \tag{5.5}$$

Now we may assume w.l.o.g. that $2y \le c_+(\tilde{L})$. (Otherwise consider "complementary" crossing changes. That is, if we apply only crossing changes at those crossings which are not in the set $\{c_1, c_2, \ldots, c_k\}$, then we have another set of crossings that satisfies all conditions in Lemma 5.2. For this set, it holds that $2y \le c_+(\tilde{L})$.)

Suppose that B_C is the maximum term of $P_L(l, m)$. Then $2(y - x) + \mu - 1 \ge w(\tilde{L}) + h$. But since the maximal degree of $P_L(l, m)$ w.r.t. l is at most $w(\tilde{L}) + s(\tilde{L}) - 1 = w(\tilde{L}) + h$ [Mo], we must have

$$2(y - x) + \mu - 1 = w(\tilde{L}) + h. \tag{5.6}$$

Furthermore, since $-(\mu - 1)$ is the minimal degree of $P_L(l, m)$ w.r.t. m [L-M], we see

$$-(\mu - 1) = c(\tilde{L}) - h - 2h_+. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain that $2(y - x) = 2c_+(\tilde{L}) - 2h_+$. This is impossible, however, since $2c_+(\tilde{L}) \ge 2h_+ + 2y$ and $x > 0$, proving Proposition 5.4.

6. Proof of Theorem A(II)

This section will be devoted to the proof of Proposition 5.3.

Suppose that some $L_{j_1,\dots,j_d}$ is not reduced. Let $L_{j_1,\dots,j_d} = \widehat{L}_1 * \widehat{L}_2 * \cdots * \widehat{L}_h$. Since $L_{j_1,\dots,j_d}$ is not reduced, some of $\widehat{L}_j$ represents trivial knots, i.e., all crossings but one of $\widehat{L}_j$ must be split. For simplicity, each of $\widehat{L}_1 \cdots \widehat{L}_\lambda$ is a trivial knot with one positive crossing and each of $\widehat{L}_{\lambda+1}, \dots, \widehat{L}_{\lambda+\nu}$ is a trivial knot with one negative crossing. Then $L_{j_1,\dots,j_d}$ is isotopic to a reduced alternating fibered link $\widehat{L}_{j_1\dots j_d} = \widehat{L}_{\lambda+\nu+1} * \cdots * \widehat{L}_h$, and hence we have

$$
\begin{aligned}
(6.1)\qquad &(1)\quad c(\widehat{L}_{j_1\cdots j_d}) = c(\widetilde{L}) - d - (\lambda+\nu),\\
&(2)\quad h(\widehat{L}_{j_1\cdots j_d}) = h(L) - (\lambda+\nu),\\
&(3)\quad h_+(\widehat{L}_{j_1\cdots j_d}) = h_+(L) - \lambda,\\
&(4)\quad h_-(\widehat{L}_{j_1\cdots j_d}) = h_-(L) - \nu,\\
&(5)\quad w(\widehat{L}_{j_1\cdots j_d}) = w(\widetilde{L}) - d + 2u_d - \lambda + \nu.
\end{aligned}
$$

Since $\widehat{L}_{j_1,\dots,j_d}$ is reduced, we can apply the induction assumption on $\widehat{L}_{j_1,\dots,j_d}$ to show that

$$B_{L_{j_1\cdots j_d}} = B_{\widehat{L}_{j_1\cdots j_d}} = (-1)^{c-d+h_- -\lambda} m^{c-h-2h_+-d+2\lambda} l^{w+h-d+2u_d-2\lambda},$$

and hence we have

$$(6.2)\qquad m^d l^{d-2u_d} B_{L_{j_1\cdots j_d}} = (-1)^{c-d+h_- -\lambda} m^{c-h-2h_+ +2\lambda} l^{w+h-2\lambda}.$$

From (6.2), we see that if $\lambda \neq 0$ then $m^d l^{d-2u_d} B_{L_{j_1\cdots j_d}}$ cannot be the maximum term of $f(l, m)$. Therefore, to prove (5.4), it is sufficient to show that $\sum_{d=1}^{k} \sum_{1\le j_1<\cdots<j_d\le k} (-1)^{c-d+h_-} = (-1)^{c+h_-+1}$, or equivalently,

$$(6.3)\qquad \sum_{d=1}^{k} \sum_{1\le j_1<\cdots<j_d\le k} (-1)^d = -1,$$

where the summation is further restricted to $\{j_1, \dots, j_d\}$ such that $L_{j_1\cdots j_d}$ has $\lambda = 0$. Thus we need to count the number of links $L_{j_1,\dots,j_d}$ with $\lambda = 0$.

If $\lambda = 0$ for all $L_{j_1\cdots j_d}$ or, in particular, if all $L_{j_1,\dots,j_d}$ are reduced, then since there are exactly $\binom{d}{k}$ choices of sequences $j_1, \dots, j_d$ from k distinct indices, $1, 2, \dots, k$, we see

$$\sum_{d=1}^{k} \sum_{1\le j_1<\cdots<j_d\le k} (-1)^d = \sum_{d=1}^{k} \binom{k}{d} (-1)^d = -1$$

and we are done. To prove (6.3) in general, let $\mathscr{S} = \{c_1, c_2, \dots, c_k\}$ be crossings found in Lemma 5.2. Since $h_+(\widetilde{L}) \neq 0 \neq h_-(\widetilde{L})$, some crossings in

$\mathscr{S}$ are positive and some are negative. For simplicity, let $L_1, \dots, L_p$ all be positive links L_i with $\alpha_i = q_i - 1$, $i = 1, 2, \dots, p$, but each of the remaining $h - p$ links $L_{p+1}, \dots, L_h$ has either negative crossings or has at most $q_i - 2$ positive crossings in $\mathscr{S}$. Note from Lemma 5.2 that $0 \le p < h$. Denote by ξ the left-hand side of equation (6.3). ξ will easily be evaluated as follows.

We see from (6.2) that $m^d l^{d-2u_d} B_{L_{j_1, \dots, j_d}}$ is the maximum term of $f(l, m)$ if and only if, to each i, $l \le i \le p$, all α_i crossings of L_i contained in $\mathscr{S}$ never occur in $\{c_{j_1}, \dots, c_{j_d}\}$. The number ξ_d of such links $L_{j_1 \cdots j_d}$ is given by

$$\xi_d = \sum_{i_1, \dots, i_p} \sum_{m_{p+1} \cdots m_h} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_p}{i_p} \binom{\alpha_{p+1}}{m_{p+1}} \cdots \binom{\alpha_h}{m_h},$$

where the summation runs over all $i_1, \dots, i_p$, $m_{p+1}, \dots, m_h$ such that $0 \le i_k \le \alpha_k - 1$, $k = 1, 2, \dots, p$ and $0 \le m_r \le \alpha_r$, $r = p+1, \dots, h$. Putting

$$\xi_0 = \binom{\alpha_1}{0} \cdots \binom{\alpha_n}{0} = 1,$$

we can write $\xi = \sum_{d=0}^{k} (-1)^d \xi_d - 1$. However, a straightforward computation shows that

$$\sum_{d=0}^{k} (-1)^d \xi_d = \left\{ \sum (-1)^{i_1 + \cdots + i_p} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_p}{i_p} \right\} \times \left\{ \sum (-1)^{m_{p+1} + \cdots + m_h} \binom{\alpha_{p+1}}{m_{p+1}} \cdots \binom{\alpha_h}{m_h} \right\}.$$

Since $h - p > 0$, the second factor is

$$\sum (-1)^{m_{p+1} + \cdots + m_k} \binom{\alpha_{p+1}}{m_{p+1}} \cdots \binom{\alpha_h}{m_h} = \prod_{i=p+1}^{h} \sum_{m_i = 0}^{\alpha_i} (-1)^{m_i} \binom{\alpha_i}{m_i} = 0,$$

and hence $\xi = -1$ as required. This proves Proposition 5.3, and the proof of Theorem A is now complete.

Theorem A immediately implies

Proposition 6.2. *For an alternating fibered link L, $\mathbf{e}(L) = w(\widetilde{L})$, where $\widetilde{L}$ is a reduced alternating diagram of L.*

7. Corollaries

We begin with a proof of Corollary 1.

Proof of Corollary 1. Since $h(\widetilde{L}) = \mathbf{b}(L) - 1$, $h(\widetilde{L})$ is an alternating-link invariant. Furthermore, it follows from Theorem 5.4 in [Mu2] that the signature

$$\begin{aligned} \sigma(L) &= -\{(c_+(\widetilde{L}) - h_+(\widetilde{L})) - (c_-(\widetilde{L}) - h_-(\widetilde{L}))\} \\ &= -\{c_+(\widetilde{L}) - c_-(\widetilde{L})\} + h_+(\widetilde{L}) - h_-(\widetilde{L}). \end{aligned}$$

Since $c_+(\widetilde{L})-c_-(\widetilde{L}) = w(\widetilde{L})$ is an alternating-link invariant [Mu5], so is $h_+(\widetilde{L})-h_-(\widetilde{L})$. Now the invariances of $h_+(\widetilde{L})$ and $h_-(\widetilde{L})$ follow immediately.

Remark 7.1. The invariances of $h_+(\widetilde{L})$ and $h_-(\widetilde{L})$ are also proved by Traczyk [Tr]. He proves that if L is a (nonsplit) alternating fibered link, then, for the homfly polynomial $Q_L(x, y, z)$ of L,

$$\begin{aligned} d^x_{\min} Q_L(x, y, z) &= -c_+(\widetilde{L}) + h_+(\widetilde{L}), \\ d^y_{\min} Q_L(x, y, z) &= -c_-(\widetilde{L}) + h_-(\widetilde{L}), \end{aligned} \tag{7.1}$$

where $d^x_{\min} Q$ and $d^y_{\min} Q$ denote the minimal degree of Q w.r.t. x and y, respectively. Since $c_+(\widetilde{L})$ and $c_-(\widetilde{L})$ are alternating-link invariants [Mu6], so are $h_+(\widetilde{L})$ and $h_-(\widetilde{L})$.

Proof of Corollary 2. Let L be an alternating fibered link and $\widetilde{L}$ its reduced alternating diagram. Then it is known [Mu2] that $\deg \Delta_L(t) = c(\widetilde{L}) - s(\widetilde{L}) + 1$. Since $\widetilde{L}$ is reduced, it follows that $\mathbf{b}(L) = h + 1 = s(\widetilde{L}) = c(\widetilde{L}) - \deg \Delta_L(t) + 1$. Q.E.D.

Next we show a few immediate consequences of the main theorems.

Proposition 7.1. *Let L_i, $i = 1, 2$, be either an alternating fibered link or a 2-bridge link. Then*

$$\mathbf{b}(L_1 \# L_2) = \mathbf{b}(L_1) + \mathbf{b}(L_2) - 1.$$

This solves Problem 10 in [B] for these links. A proof follows from the fact that $P_L(l, m)$ is multiplicative w.r.t. a connected sum.

Proposition 7.2. *Let L be a 2-bridge link of type (α, η), where $0 < \beta < \alpha$ and β is odd. Then*

$$\begin{aligned} &(1)\quad \mathbf{b}(L) = 2 \quad \textit{iff } \beta = 1. \\ &(2)\quad \mathbf{b}(L) = 3 \quad \textit{iff either} \end{aligned} \tag{7.2}$$

(i) *for some $p, q > 0$, $\alpha = 2pq + 3p + 3q + 4$ and $\beta = 2p + 3$, or*
(ii) *for some $p > 0$, $\alpha = 2pq + p + q + 1$ and $\beta = 2p + 1$.*

Proof. A proof of (1) is obvious. To show (2), we note by Theorem B that $\mathbf{b}(L) = 3$ iff the continued fraction form of L is either (1) $t = 1$ and all $n_{1j} = 1$ but one $n_{1k} = 2$, i.e. $[2, 2, \dots, 2, 4, 2, \dots, 2]$, or (ii) $t = 2$ and $n_{ij} = 1$ for all i and j, i.e., $[2, 2, \dots, 2, -2, -2, \dots, -2]$. A simple computation shows that, for each case, α and β have the form given in Proposition 7.2.

Remark 7.2. Proposition 7.2 proves the conjecture by Przytycki [P].

We call an n-braid $\gamma = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k}$, $\varepsilon_i = \pm 1$, $1 \le i_1, \dots, i_k \le n-1$, a *homogeneous* braid if $\varepsilon_j = \varepsilon_l$ whenever $i_j = i_l$. γ is called an *alternating* braid if $\varepsilon_j = \varepsilon_l$ iff $i_j \equiv i_l \pmod 2$. γ is called *reduced* if each σ_j, $1 \le j \le n-1$, appears at least twice in γ.

Proposition 7.3. *If γ is an alternating and reduced n-braid, then the braid index of the closure $\hat{\gamma}$ of γ is exactly n.*

Proof. γ is an alternating fibered link with a natural downward orientation. Since γ is reduced, $\hat{\gamma}$ has exactly n (nontrivial) $*$-components, and hence Proposition 7.3 follows from Theorem A.

The closure of a homogeneous n-braid may have a smaller braid index than n, but if it has the braid index n, then we can determine the minimal number of crossings of its closure. In fact, let $\mathbf{c}(L)$ denote the minimal number of crossings among all diagrams that a link L can have. Then we can prove

Proposition 7.4. *Let γ be a homogeneous n-braid. Let d_i be the exponent sum of σ_i in γ. If $\mathbf{b}(\hat{\gamma}) = n$ or if γ is a reduced pure n-braid, then $\mathbf{c}(\hat{\gamma}) = \sum_{i=1}^{n-1} |d_i|$.*

Proof. If γ is reduced and pure, then $\mathbf{b}(\hat{\gamma}) = n$ by Proposition 2.2. Therefore, we assume that $\mathbf{b}(\hat{\gamma}) = n$. Let $d = \sum_{i=1}^{n-1} |d_i|$ and let $\widetilde{L}$ be any diagram of L. Then the degree of the reduced Alexander polynomial $\Delta(t)$ of $\hat{\gamma}$ is bounded above by $c(\widetilde{L}) - s(\widetilde{L}) + 1$. On the other hand, since γ is homogeneous, $\deg \Delta(t) = c(\tilde{\gamma}) - s(\tilde{\gamma}) + 1 = d - n + 1$, where $\tilde{\gamma}$ is the diagram obtained from $\hat{\gamma}$ as a closed n-braid. Therefore $c(\widetilde{L}) - s(\widetilde{L}) + 1 \geq d - n + 1$ and hence $c(\widetilde{L}) - d \geq s(\widetilde{L}) - n \geq 0$ by Proposition 2.1. Therefore $c(\widetilde{L}) \geq d$, i.e., $d = \mathbf{c}(\hat{\gamma})$.

As a typical application, we have

Proposition 7.5. *Let $L(q, r)$ be the torus link of type (q, r), where $2 \leq q \leq r$. Then $\mathbf{c}(L(q, r)) = r(q-1)$.*

Proof. The minimal bridge number of a link is at most equal to its braid index. Since the minimal bridge number of $L(q, r)$ is q [S], it follows that $\mathbf{b}(L(q, r)) = q$. (This fact will also follow from [J, Theorems 9.7 and 15.1] for a torus *knot* of type (q, r).) Since $L(q, r)$ is represented as the closure of a homogeneous q-braid $(\sigma_1 \sigma_2 \cdots \sigma_{q-1})^r$, the proposition follows immediately.

8. Special alternating links

To determine the braid index of an alternating link L, we may need the braid index of each $*$-component of L, which is a *special* alternating link. Generally, there is no known algorithm which can be used to determine the braid index of a special alternating link. However, there are some particular types of special alternating links besides 2-bridge links and fibered links, whose braid indices are easily determined.

Let L be a special alternating link and $\widetilde{L}$ a reduced special alternating diagram. Let $\Gamma(\widetilde{L})$ be the weighted graph associated with $\widetilde{L}$. Since $\widetilde{L}$ is special alternating, we may assume that Γ is positive and even. A (planar) graph Γ (or the link diagram $\widetilde{L}$ associated with Γ) is said to be of the *nonmultiple* type if no two vertices in the dual graph Γ^* of Γ are connected by a single edge.

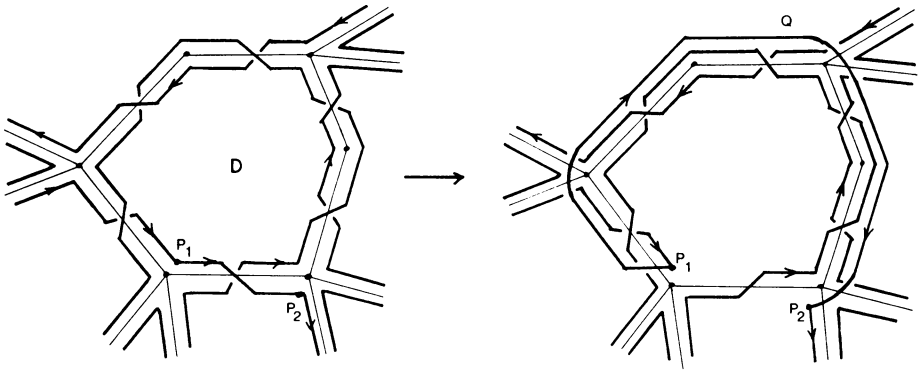


FIGURE 8

Theorem 8.1. *Let L be a special alternating link and $\widetilde{L}$ a reduced special alternating diagram. Let $s(\widetilde{L})$ denote the number of Seifert circles in $\widetilde{L}$. Then $\mathbf{b}(L) = s(\widetilde{L})$ iff $\Gamma(\widetilde{L})$ is of the nonmultiple type.*

The "if" part of the theorem was originally stated by K. Kobayashi in [K2]. However, we will prove both implications.

Proof of Theorem 8.1. First we prove the "only if" part. Suppose that $\Gamma(\widetilde{L})$ is not of the nonmultiple type. Then it suffices to prove that L has a diagram $\widehat{L}$ such that $s(\widehat{L}) < s(\widetilde{L})$. In fact, then, by Proposition 2.1, $\mathbf{b}(L) \leq s(\widehat{L}) < s(\widetilde{L})$.

Since $\Gamma(\widetilde{L})$ is not of the nonmultiple type, there are two vertices in the dual graph Γ^* of $\Gamma(\widetilde{L})$ which are joined by a single edge. Therefore, there is an edge e in Γ which connects two nontwigs, say v_1 and v_2, where a twig is a vertex with valency two. Let D be the domain in $S^2 - \Gamma$ whose boundary contains e. D corresponds to one shaded domain in $S^2 - \widetilde{L}$. See Figure 8(a).

Take two points P_1 and P_2 on an overcrossing arc that passes over the crossing associated with e. Then replace the short arc $\widehat{P_1P_2}$ by the long path $\widehat{P_1QP_2}$ depicted in Figure 8(b). $\widehat{P_1QP_2}$ is an over-crossing arc which lies outside D but is close to the boundary of D. It is easy to see from Figure 8(b) that a new diagram $\widehat{L}$ has $s(\widetilde{L}) - 1$ Seifert circles. This proves the "only if" part of Theorem 8.1. Therefore, if $\Gamma(\widetilde{L})$ is not of the nonmultiple type, then, by (2.3),

$$l\text{-span}\, P_L(l, m) < 2\{s(\widetilde{L}) - 1\}. \tag{8.1}$$

For the "if" part, suppose that $\Gamma(\widetilde{L})$ is of the nonmultiple type. Then, by Proposition 2.3, it is enough to prove

$$l\text{-span}\, P_L(l, m) = 2\{s(\widetilde{L}) - 1\}. \tag{8.2}$$

We evaluate $a_l(L)$ and $b_l(L)$; $a_l(L)$ is easily evaluated even for positive links.

Proposition 8.2. *Let L be a (nonsplit) positive link. Let $r = \deg \Delta_L(t)$. Then $A_L = (-1)^r m^r l^r$.*

For a positive link L, it it is shown [K1] that $r = a_l(L)$ and r is also the maximal degree of m in $P_L(l, m)$. Since L is a positive link, almost the same argument employed in [Mu3] can be used to prove Proposition 8.2 by induction on $c(\widetilde{L})$. What we need to prove (8.2) is the fact that $r = a_l(L)$, which is already proved in [K1], and hence we omit a proof. For details, see [K1] or [Mu3].

Now if $\widetilde{L}$ is reduced special alternating diagram of L, then $r = c(\widetilde{L}) - s(\widetilde{L}) + 1$, and hence (8.2) will follow from

$$b_l(L) = c(\widetilde{L}) + s(\widetilde{L}) - 1. \tag{8.3}$$

We will prove (8.3) by induction on $c(\widetilde{L})$. We may assume w.l.o.g. that $\widetilde{L}$ is positive.

If $c(\widetilde{L}) = 2$, then Γ is  and (8.3) obviously holds.

Suppose that two nontwigs v_1 and v_2 are connected by a sequence of twigs $w_1, w_2, \dots, w_k$, $k \geq 1$.

Let e_i be an edge joining w_i and w_{i+1}, $i = 0, 1, \dots, k$, where $w_0 = v_1$ and $w_{k+1} = v_2$. Apply the formula (2.2) on a crossing associated with the first edge e_1. For convenience, we denote the polynomials $P(l, m)$ of L, L_- and L_0 by P_{k+1}, P_{k-1} and P_k, respectively, since these links contain exactly $k+1$, $k-1$ and k edges connecting v_1 and v_2. Now (2.2) implies that $-P_{k+1}(l, m) = lmP_k(l, m) + l^2 P_{k-1}(l, m)$. Repeated applications of (2.2) yield, for $k - n \geq 0$,

$$P_{k+1}(l, m) = q_n(l, m)P_{k-n+1}(l, m) + r_n(l, m)P_{k-n}(l, m), \tag{8.4}$$

where q_n and r_n are polynomials in $l^{\pm 1}$ and $m^{\pm 1}$. Then an easy induction on n will prove

Lemma 8.3. *If $k - n \geq 0$, then $B_{q_n} = \pm l^n m^n$ and $B_{r_n} = \pm l^{n+1} m^{n-1}$.*

Proof. Since $q_1(l, m) = -lm$ and $r_1(l, m) = -l^2$, the lemma holds trivially for $n = 1$. Now, from (8.4), we have

$$\begin{aligned} P_{k+1}(l, m) &= q_n(l, m)(-lmP_{k-n}(l, m) - l^2 P_{k-n-1}(l, m)) \\ &\quad + r_n(l, m)P_{k-n}(l, m) \\ &= \{-lmq_n(l, m) + r_n(l, m)\}P_{k-n}(l, m) \\ &\quad - l^2 q_n(l, m)P_{k-n-1}(l, m) \end{aligned}$$

and hence,

$$q_{n+1}(l, m) = -lmq_n(l, m) + r_n(l, m) \quad \text{and} \quad r_{n+1}(l, m) = -l^2 q_n(l, m).$$

Since, by the induction assumption, $B_{q_n} = \pm l^n m^n$ and $B_{r_n} = \pm l^{n+1} m^{n-1}$, we have $B_{q_{n+1}} = \pm m^{n+1} l^{n+1}$ and $B_{r_{n+1}} = \pm m^n l^{n+2}$. This completes the induction.

Now (8.4) and Lemma 8.3 imply that

$$P_{k+1}(l, m) = q_k(l, m)P_1(l, m) + r_k(l, m)P_0(l, m),$$

where $B_{q_k} = \pm l^k m^k$ and $B_{r_k} = \pm m^{k-1} l^{k+1}$. We should note that L_0 is of the nonmultiple type or the connected sum of these, while L_1 is not of the nonmultiple type. Since $b_l(L)$, $c(\widetilde{L})$ and $s(\widetilde{L})$ are additive w.r.t. a connected sum, by induction we have $b_l(P_o(l, m)) = c(\widetilde{L}_0) + s(\widetilde{L}_0) - 1$. Since $c(\widetilde{L}_0) = c(\widetilde{L}) - k - 1$ and $s(\widetilde{L}_0) = s(\widetilde{L})$, it follows that $b_l(P_0(l, m)) = c(\widetilde{L}) + s(\widetilde{L}) - k - 2$ and hence $b_l(r_k(l, m)(P_0(l, m)) = c(\widetilde{L}) + s(\widetilde{L}) - 1$. On the other hand, L_1 is not of the nonmultiple type, and hence (8.1) shows that $b_l(P_1(l, m)) < c(\widetilde{L}_1) + s(\widetilde{L}_1) - 1$. Since $c(\widetilde{L}_1) = c(\widetilde{L}) - k$ and $s(\widetilde{L}_1) = s(\widetilde{L})$, it follows that $b_l(q_k(l, m)P_1(l, m)) < c(\widetilde{L}) + s(\widetilde{L}) - 1 - k + k = c(\widetilde{L}) + s(\widetilde{L}) - 1$, and hence $b_l(P_{k+1}(l, m)) = b_l(r_k(l, m)P_0(l, m)) = c(\widetilde{L}) + s(\widetilde{L}) - 1$, proving (8.3).

Remark 8.1. It is not difficult to determine $b_m(L)$ for a prime special alternating link L of the nonmultiple type. In fact, if there are exactly q nontwigs in $\Gamma(\widetilde{L})$, then $b_m(L) = c(\widetilde{L}) - 3(s(\widetilde{L}) - 1) - 2(q - 1)$.

Example 8.1. Let L be a pretzel link L of type $(a_1, a_2, \dots, a_{2k})$, where $a_i \geq 2$ for all i. Then the orientation of L can be chosen so that L is a special alternating link of the nonmultiple type. Therefore $\mathbf{b}(L) = 2k$.

Appendix. Proof of Lemma 5.2. Let L be an alternating fibered link and $\widetilde{L} = L_1 * \cdots * L_h$ a *-product representation of L. L_i is an elementary torus link. L_i spans an orientable surface F_i consisting of two disks $D_{i,1}$ and $D_{i,2}$ which are joined by finitely many bands. Each band is half twisted positively or negatively. L is constructed from these L_i by a sequence of identifications of two disks $D_{i,1}$ (or $D_{i,2}$) and $D_{j,1}$ or $D_{j,2}$ [Mu1]. To illustrate this construction, we use a *construction* graph defined as follows. Take two vertices $v_{i,1}$ and $v_{i,2}$ on the plane associated with the disks $D_{i,1}$ and $D_{i,2}$ of F_i. $v_{i,1}$ and $v_{i,2}$ are joined by an edge e_i. If $D_{i,1}$, say, of F_i is identified to $D_{j,1}$, say, of F_j, then $v_{i,1}$ and $v_{j,1}$ associated with $D_{i,1}$ and $D_{j,1}$ are identified. The resulting graph $\Sigma(L)$ associated with L is not only planar, but it is also a tree, since L is fibered. (For more details, see [Mu1].)

Now to construct a descending diagram L_0 from L, first define an order to the components $K_1, K_2, \dots, K_\mu$ of L and pick up a base point b_i from each component K_i. Let L_0 be a descending diagram of L w.r.t. this order and base points. L_0 represents a trivial link with trivial components $K_1', \dots, K_\mu'$. L_0 is obtained from L by applying crossing changes at, say, $c_1, \dots, c_g$. Let $\mathcal{S} = \{c_1, c_2, \dots, c_g\}$. If $\mathcal{S}$ satisfies Lemma 5.2, we are done. If not, two cases can occur.

Case 1. None of the crossings on some L_j belongs to $\mathcal{S}$.

Case 2. All crossings on some L_j belong to $\mathcal{S}$.

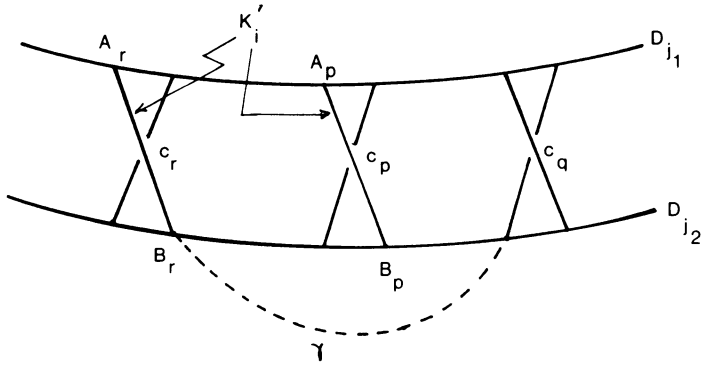


FIGURE 9

Since Case 2 is *dual* to Case 1, we consider only Case 1.

Let K_i' be the *first* component of L_0 that passes over crossings in L_j. Since no crossings on L_j are in $\mathscr{S}$, no preceding component K_j', $j < i$, passes under crossings in L_j. Let c_r be the *first* crossing on L_j over which K_i' passes.

Suppose that K_i' crosses over another crossing c_p on L_j, see Figure 9. Then K_i' must have a path γ (in L_0) joining B_r to A_p. Since the construction graph Σ is a tree, γ must cross under some crossing on L_j. Let c_q be the first crossing on L_j under which γ passes. If $c_q \neq c_r$, then it follows from the choice of K_i' that a crossing change must be made at c_q and hence $c_q \in \mathscr{S}$, a contradiction. Therefore, $c_q = c_r$, i.e., c_r is a self-crossing of K_i'. Then we make an unnecessary crossing change at c_r without violating the property that L_0 be a trivial link. Now $\mathscr{S}' = \mathscr{S} \cup \{c_r\}$ satisfies the condition in Lemma 5.2 for L_j.

Repeat this argument for L_t when none of the crossings on L_t belong to $\mathscr{S}$. This proves Lemma 5.2.

Added in proof. Conjecture A is false. A counterexample can be found in [M-P].

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